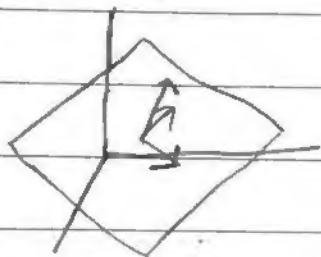


## - lecture 4 § 12.4 - cross product

- our goal: to answer a geometric question



• given 2 vectors: (in  $\mathbb{R}^3$ )

$$\vec{u} = \langle u_1, u_2, u_3 \rangle \text{ and } \vec{v} = \langle v_1, v_2, v_3 \rangle$$

construct a vector

$\vec{w} = \langle w_1, w_2, w_3 \rangle$  such that  
 $\vec{w}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$

- How? we know 
$$\begin{cases} 0 = \vec{u} \cdot \vec{w} = u_1 w_1 + u_2 w_2 + u_3 w_3 \\ 0 = \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \end{cases}$$

lets eliminate  $w_3$

① multiply eq① by  $v_3$  and eq② by  $u_3$  to obtain:

$$\begin{cases} 0 = v_3(\vec{u} \cdot \vec{w}) = (u_1 v_3) w_1 + (u_2 v_3) w_2 + (u_3 v_3) w_3 \\ 0 = u_3(\vec{v} \cdot \vec{w}) = (u_3 v_1) w_1 + (u_3 v_2) w_2 + (u_3 v_3) w_3 \end{cases}$$

② Subtract eq② from eq① to get:

$$0 = v_3(\vec{u} \cdot \vec{w}) - u_3(\vec{v} \cdot \vec{w})$$

$$= (u_1 v_3 - u_3 v_1) w_1 + (u_2 v_3 - u_3 v_2) w_2$$

$$\text{eq③: } -(- (u_1 v_3 - u_3 v_1)) w_1 + (u_2 v_3 - u_3 v_2) w_2$$

$$\begin{aligned} -ax + by &= 0 & \text{if } x=b \text{ and } y=a \\ -ab + ba &= 0 \end{aligned}$$

hence eq (3) has a solution

$$\begin{cases} w_1 = u_2 v_3 - u_3 v_2 \\ w_2 = -(u_1 v_3 - u_3 v_1) \end{cases}$$

• inputting these into eq (1), we obtain:

$$0 = u_1 w_1 + u_2 w_2 + u_3 w_3$$

$$0 = u_1 (u_2 v_3 - u_3 v_2) + u_2 (- (u_1 v_3 - u_3 v_1)) + u_3 v_3$$

$$0 = \cancel{u_1 u_2 v_3} - u_1 u_3 v_2 - \cancel{u_1 u_2 v_3} + u_2 u_3 v_1 + u_3 v_3$$

$$u_3 (u_2 v_1 - u_1 v_2) + u_3 v_3$$

Hence either  $u_3 = 0$  or  $w_3 = u_1 v_2 - u_2 v_1$

- Claim: (ignoring the fact that  $u_3$  may be 0) we have a solution:

$$\vec{w} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

the check is left to u.s.

- Def (Determinant): the determinant of the  $2 \times 2$   $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  matrix is  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

• The determinant of a  $3 \times 3$  matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

$$\det \begin{bmatrix} -1 & 3 & 7 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$-1 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 7 \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$-1(-1(1) + 1(0)) - 3(0(1) - 1(1)) + 7(0(1) - (-1)(1))$$

$$-1(-1 + 0) - 3(0 - 1) + 7(+1)$$

$$1 + 3 + 7 = 11$$

- cross product definition:

• let  $\vec{u} = \langle u_1, u_2, u_3 \rangle$ ,  $\vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

• cross product of  $\vec{u}$  with  $\vec{v}$  is

$$\vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2) \vec{i} - (u_1 v_3 - u_3 v_1) \vec{j} + (u_1 v_2 - u_2 v_1) \vec{k}$$

$$= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

- This has all been done in  $\mathbb{R}^3$ .....

• This cross product only works in  $\mathbb{R}^3$ .....

• The cross product is a vector operation.

(vector in  $\mathbb{R}^3$  x vector in  $\mathbb{R}^3 \mapsto$  vector in  $\mathbb{R}^3$ )

- Algebraic properties of  $\times$  product is

Let  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  and  $c \in \mathbb{R}$

$$\textcircled{1} \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$\textcircled{2} (c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$$

$$\textcircled{3} \vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

$$\textcircled{4} (\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$$

$$\textcircled{5} \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$\textcircled{6} \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

- Geometric properties of cross product

Let  $\vec{u}, \vec{v} \in \mathbb{R}^3$

①  $\vec{u} \times \vec{v}$  is orthogonal to both  $\vec{u}$  and  $\vec{v}$

②  $|\vec{u} \times \vec{v}|$  - magnitude - is  $|\vec{u}||\vec{v}|\sin(\theta)$  where  $\theta$  is the angle between them.

③  $\vec{u} \times \vec{v} = \vec{0}$  iff  $\vec{u}$  is parallel to  $\vec{v}$ .